香港中文大學
答題簿
THE CHINESE UNIVERSITY OF HONG KONG
ANSWER BOOK

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姓名(中文) Name (Chinese) .... (英文) (English) ....

系別 Department .... 年級 Year .... 學號 Student ID ....

科目 Course ....

1. Find the solution of the given initial data

$$y'' + 4y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = -1.$$

And sketch the graph of the solution and describe its behavior as t increases.

Solution: Let $y = ce^{rt}$ be the solution with c, r to be determined.
Then r satisfies:

$$r^2 + 4r + 3 = 0 \qquad 5'$$

The roots of the above characteristic equation:

$$r_1 = -3, \quad r_2 = -1. \qquad 5'$$

So the general solution to the homogeneous equation is:

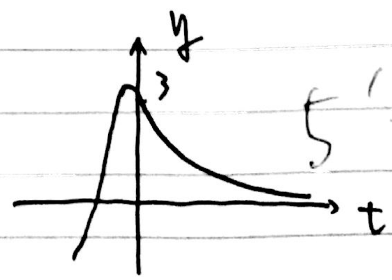
$$y(t) = C_1 e^{-3t} + C_2 e^{-t}, \qquad 5'$$

where C_1, C_2 are constants.
Substitute the initial conditions.

$$\begin{cases} C_1 + C_2 = 3 \\ -3C_1 - C_2 = -1 \end{cases} \Rightarrow \begin{cases} C_1 = -1 \\ C_2 = 4 \end{cases} \qquad 5'$$

Hence the particular solution is:

$$y(t) = -e^{-3t} + 4e^{-t}$$



$5'$

$y(t) \to 0$ as $t \to \infty$

$y'(t) = 3e^{-3t} - 4e^{-t} = e^{-t}(3e^{-2t} - 4) < 0$ for t sufficiently large. $(t \geq 0)$

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2. Determine the longest interval in which the given initial value problem is certain to have a unique ~~twice~~-differentiable solution. Do not attempt to find the solution.

$$(t-1)y'' - 3t\,y' + 5y = \sin t, \quad y(-3)=2, \quad y'(-3)=1.$$

Solution:

$$y'' - \frac{3t}{t-1}y' + \frac{5}{t-1}y = \frac{\sin t}{t-1}.$$

twice ~~solution~~ differentiable solution exists for interval where

$$-\frac{3t}{t-1}, \quad \frac{5}{t-1}, \quad \frac{\sin t}{t-1} \quad \text{are continuous.}$$

5'

They continuous at: $(-\infty, 1)$ and $(1, \infty)$

Considering the initial values, the longest interval is

5'

$$(-\infty, 1).$$

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3. Find the solution of the given initial value problem

$$y'' - 2y' + 5y = 0, \quad y(\tfrac{\pi}{2}) = 0, \quad y'(\tfrac{\pi}{2}) = 4.$$

Sketch the graph of solution and describe its behavior for increasing t.

Solution Suppose $y(t) = Ce^{rt}$ is the solution with C, r to be determined.

Characteristic equation:

$$r^2 - 2r + 5 = 0.$$ 5'

The roots are :

$$r_1 = 1+2i, \quad r_2 = 1-2i.$$ 5'

General solution:

$$y(t) = e^t(C_1 \sin 2t + C_2 \cos 2t)$$ 5'

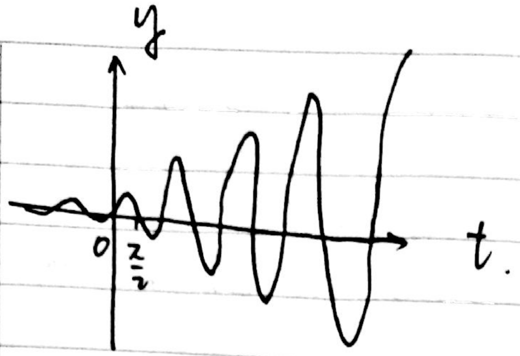
Substitute the initial conditions:

$$\begin{cases} -e^{\frac{\pi}{2}} C_2 = 0. \\ -2e^{\frac{\pi}{2}} C_1 = 4 \end{cases} \Rightarrow \begin{cases} C_1 = -2e^{-\frac{\pi}{2}}. \\ C_2 = 0 \end{cases}$$ 5'

~~Part~~ So the solution to the initial value problem:

$$y(t) = -2e^{t-\frac{\pi}{2}} \sin 2t.$$

$\sin 2t$ is a periodic function while $e^{t-\frac{\pi}{2}}$ increases as t increases. Hence $y(t)$ is a function of fixed frequency with amptitude increases as t increases.



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4. Solve the given initial ~~data~~ value problem

$$y'' - 6y' + 9y = 0. \quad y(0) = 0, \; y'(0) = 3.$$

Sketch the graph of the solution and describe its behavior as t increases.

Solution: Let $y = Ce^{rt}$ is the solution.

Characteris equation:

$$r^2 - 6r + 9 = 0.$$ 5'

root: $r_{1,2} = 3$. (repeated) 5'

General solution:

$$y(t) = C_1 e^{3t} + C_2 t e^{3t}.$$ 5'

Substitute the initial conditions,

$$\begin{cases} C_1 = 0 \\ 3C_1 + C_2 = 3 \end{cases} \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = 3. \end{cases}$$ 5'

So the solution for the initial value problem:

$$y(t) = 3te^{3t}.$$

$$y' = 3e^{3t} + 9te^{3t} = 3e^{3t}(1+3t) = \begin{cases} < 0 & t < -\frac{1}{3} \\ > 0 & t > -\frac{1}{3} \end{cases}$$

So $y(t)$ decreases on $(-\infty, -\frac{1}{3})$,
increases on $(-\frac{1}{3}, \infty)$ a

$y(t)$ goes to $+\infty$ as $t \to +\infty$.
goes to 0 as $t \to -\infty$.

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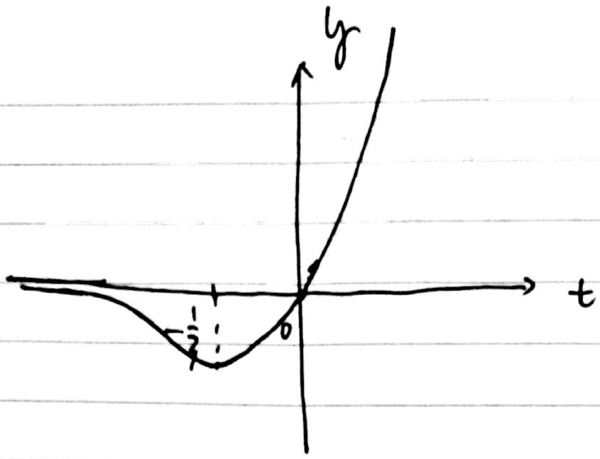

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5. Find a second independent solution of the equation given by using Abel's formula.

$$(x-1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$$

Solution: Suppose y_2 is the second independent solution

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2. \qquad 5'$$

$$y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$$

Let $p(x) = \frac{-x}{x-1}$, $Q(x) = \frac{1}{x-1}$.

Abel's formular,

$$W' + pW = 0.$$

$$W(x) = W_0 e^{-\int p(x)dx} = W_0 e^{\int \frac{x}{x-1} dx} \qquad 5'$$

$$= W_0 e^{x + \ln(x-1)} = W_0 e^x (x-1) \qquad (x>1)$$

$$\Rightarrow y_2' - y_2 = x-1$$

$$(y_2 e^{-x})' = (x-1)e^{-x}$$

$$\Rightarrow e^{-x} y_2(x) = -xe^{-x}$$

$$y_2(x) = -x \qquad 5'$$

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